Aim Classify all ECs.

Idea Find space $\mathcal{M}$ "parameter space"

+ EC $\mathcal{E}/\mathcal{M}$ "universal elliptic curve"

s.th. $\forall$ E/S there is a unique

$$u : S \to \mathcal{M} \quad \text{s.th.} \quad E \cong u^* \mathcal{E} := S \times_{\mathcal{M}} \mathcal{E}.$$

$\mathcal{M}$ only exists as **stack** since elliptic curves have automorphisms $\neq$ id. (e.g. $[-1]$)

Example Consider over $S = \operatorname{Spec} \mathbb{Q}$

$$E : y^2 = f(x) \quad , \quad E' : \; Dy^2 = f(x) \quad D \notin (\mathbb{Q}^\times)^2$$

$$E \not\cong E' \quad \text{but} \quad E_{\mathbb{Q}(\sqrt{D})} \cong E'_{\mathbb{Q}(\sqrt{D})}$$

Namely recall that when picking $x \in H^0(E, \mathcal{O}(2e))$

$y \in H^0(E, \mathcal{O}(3e))$

for a Weierstraß eqn, these are unique up to

$x \mapsto ux + v \quad u \in k^\times$

$y \mapsto py + qx + r \quad p \in k^\times$

and it is not possible to transform $y^2 - f(x)$ to $Dy^2 - f(x)$ by this substitution.

Over $\mathbb{Q}(\sqrt{D})$ on the other hand, simply take $y \mapsto \sqrt{D}\, y$.

If $\mathcal{M}$ were a scheme, there would be unique

$u, u' \in \mathcal{M}(\mathbb{Q})$ s.t. $u^*\mathcal{E} \cong E$, $(u')^*\mathcal{E} \cong E'$.

But $u = u'$ in $\mathcal{M}(\mathbb{Q}(\sqrt{D}))$ & $\mathcal{M}(\mathbb{Q}) \hookrightarrow \mathcal{M}(\mathbb{Q}(\sqrt{D}))$

since $E_{\mathbb{Q}(\sqrt{D})} \cong E'_{\mathbb{Q}(\sqrt{D})}$ so this is impossible.

Deeper reason here $\mathrm{Aut}(E) \supseteq \{\pm 1\} \neq \{1\}$.

E' arises by étale descent of $E_{\mathbb{Q}(\sqrt{D})}$ along

$\mathrm{Spec}\, \mathbb{Q}(\sqrt{D}) \to \mathrm{Spec}\, \mathbb{Q}$ for "twisted Galois action".

(The family $Dy^2 = f(x)$ are quadratic twists of E.)

This is analogous to argument that $S \mapsto \mathrm{Pic}(S)$

cannot be representable, only with gluing for étale topology

instead of Zariski topology.

Will see Variants of $\mathcal{M}$ exist as schemes.

3 ways to construct
- Equations (Igusa)
- GIT (Mumford)
- Functor criterion (Grothendieck, Artin)

§1 The result

Thm

$\tilde{\mathcal{M}}[\frac{1}{6}] : Sch/_{\mathbb{Z}[\frac{1}{6}]} \longrightarrow Set$

$S \longmapsto \left\{ (E, \pi) \;\middle|\; \begin{array}{l} E/S \text{ EC} \\ \pi \in \Gamma(E, \Omega^1_{E/C}) \\ \text{global generator} \end{array} \right\} / \cong$

is representable by the affine scheme

$\operatorname{Spec} \underbrace{\mathbb{Z}[\frac{1}{6}][a, b][\Delta^{-1}]}_{=: R} \quad \Delta = 4a^3 + 27b^2,$

with universal pair

$\mathcal{E} = V_+(Y^2 Z - X^3 - aXZ^2 - bZ^3) \subseteq \mathbb{P}^2_R$.

π = unique section of $\Omega^1_{\mathcal{E}/R}$ s.th.

$\pi|_{D_+(Z) \cap \mathcal{E}} = \frac{-dx}{2y} = \frac{-dy}{3x^2 + a}$

On $D(y)$ — On $D(3x^2 - a)$ — Contains $V(y) \cap \mathcal{E}$ since $x^3 + ax + b$ is separable.

Agree on intersection because

$0 = d(y^2 - x^3 - ax - b) = 2y\,dy - (3x^2 + a)\,dx.$

Explanations 1) $(E,\pi) \cong (E',\pi')$

$$\underset{def}{=} \exists f: E \xrightarrow{\cong} E' \text{ s.t. } f^*\pi' = \pi.$$

2) π is a global generator $\iff$ $\pi(s) \in \Omega^1_{E(s)/k(s)}$ is global gen $\forall s \in S$.

3) Implicit in Thm is that a pair (E,π) has no automorphisms if $6 \in \mathcal{O}_S^\times$.

If $2=0$ in $\mathcal{O}_S$, then any pair has auto $[-1]$ since $[-1]^*\pi = -\pi = \pi$.

$\longrightarrow$ Method cannot extend to char 2.

4) For any $(E,\pi) \in \tilde{\mathcal{M}}(S)$, $\Omega^1_{E/S} \cong \mathcal{O}_E$, which provides the obstruction for some E/S to occur in $\tilde{\mathcal{M}}$.

Since $\Omega^1_{E/S} = p^*e^*\Omega^1_{E/S}$, any family E occurs Zariski locally in $\tilde{\mathcal{M}}$.

§2 Proof of Thm

$p : E \to S$ with section $e : S \to E$ AV Lect 8 : e is closed immersion

$e(S) =: V(\mathcal{I})$ Cartier divisor

(i.e. $\mathcal{I} = \mathcal{O}_U \cdot f$ locally on E w/ f non-zero div)

We know

·) $\mathcal{O}(3e)$ relatively very ample since fiberwise so

·) $p_* \mathcal{O}(3e)$ v.b. of rank 3 on S.

$\Longrightarrow \quad E \hookrightarrow \mathbb{P}\left((p_* \mathcal{O}(3e))^\vee \right)$ closed embed. into twist of $\mathbb{P}^2_S$.

How to make this more explicit?

Consider $\mathcal{O}_E \subset \mathcal{O}_E(e) \subset \mathcal{O}_E(2e) \subset \cdots$

Lem 1) $\omega_E := e^* \Omega^1_{E/S} \cong \mathcal{I}/\mathcal{I}^2$ (Hodge bundle)

2) $\forall\, n \in \mathbb{Z} \quad \mathcal{O}(ne) / \mathcal{O}((n-1)e) \cong \omega_E^{\otimes(-n)}$.

Proof 1)

$$\begin{array}{ccc} S & \xrightarrow{e} & E \\ & \searrow^{=} & \downarrow p \\ & & S \end{array}$$

given $\mathcal{I}/\mathcal{I}^2 \longrightarrow e^*\Omega^1_{E/S} \longrightarrow \underbrace{\Omega^1_{S/S}}_{=0} \longrightarrow 0$

$0 \nearrow$

Left exactness because S/S is smooth.

$$\Longrightarrow \mathcal{I}/\mathcal{I}^2 \cong e^*\Omega^1_{E/S}.$$

2) Have $\quad 0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_E \longrightarrow \mathcal{O}_{e(S)} \longrightarrow 0$

$\mathcal{I}^{-n} \otimes -$: $\quad 0 \longrightarrow \mathcal{O}((n-1)e) \longrightarrow \mathcal{O}(ne)$

$$\longrightarrow \mathcal{I}^{-n} \otimes \mathcal{O}_{e(S)} \longrightarrow 0$$

$$\parallel$$

$$(\mathcal{I}/\mathcal{I}^2)^{\otimes -n} = \omega_E^{-n}.$$

$\square$

<u>Recall</u> §4 $\quad p_*\mathcal{O}(ne)$ is rk n $\quad n \geq 1$

vector bundle on S.

& $\quad R^1p_*\mathcal{O}(ne) = 0$

<u>Thus</u>

$$0 \longrightarrow p_*\mathcal{O}((n-1)e) \longrightarrow p_*\mathcal{O}(ne) \longrightarrow \omega_E^{\otimes -n} \longrightarrow 0$$

is exact for $n \geq 2$,

while $p_*\mathcal{O}_E = \mathcal{O}_S \xrightarrow{\cong} p_*\mathcal{O}_E(e)$

since $R^1p_*\mathcal{O}_E$ is a line bundle.

Thus get filtration by vector bundles

$$0 \subset \mathcal{O}_S \subset p_*\mathcal{O}(2e) \subset p_*\mathcal{O}(3e) \subset \cdots \subset p_*\mathcal{O}(ne)$$

w/ gradeds $\mathcal{O}_S, \omega^{\otimes -2}, \omega^{\otimes -3}, \ldots$

Assume $\omega = \mathcal{O}_S \cdot \pi$ trivial. (Note that such π gives $p^*\pi \in \Gamma(E, \Omega^1_{E/S})$ generator.)

$$\Longrightarrow \omega^{\otimes i} = \mathcal{O}_S \cdot \pi^i.$$

On small enough $U \subseteq S$ @ $\exists\ x \in p_*\mathcal{O}(2e)$ s.t.

$$(x \mod \mathcal{O}_S) = \pi^{-2}$$

& $y \in p_*\mathcal{O}(3e)$ s.t.

$$(y \mod p_*\mathcal{O}(3e)) = \pi^{-3}.$$

Note that this forces $(1, x)$ resp. $(1, x, y)$ to be bases of $p_*\mathcal{O}(2e)$ resp. $p_*\mathcal{O}(3e)$.

@ e.g. affine

Note that $1, x, y$ give $\mathbb{P}(p_* \mathcal{O}(3e)^\vee) \cong \mathbb{P}^2_U$

and realize $E|_U$ in Weierstraß form.

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6 .$$

This is <u>the</u> Weierstraß eq for choice $1, x, y$.

a_i are determined uniquely from linear dependence of

$$1, x, y, x^2, xy, y^2, x^3 \text{ in } p_* \mathcal{O}(6e).$$

<u>Conversely</u> Giving E in such Weierstraß form, coordinates x, y provide sections of $\mathcal{O}(2e)$, $\mathcal{O}(3e)$ of considered type. (In partic, $p_*\mathcal{O}(2e)$, $p_*\mathcal{O}(3e)$ free)

Let (π', x', y') be other choice. Then

$$\pi = u^{-1}\pi'$$

$$x = u^2 x' + r$$

$$y = u^3 y' + s u^2 x' + t$$

(Following Deligne "Courbes Elliptique : Formulaire" here)

for uniquely determined $u \in \mathcal{O}_S^\times$, $r, s, t \in \mathcal{O}_S$.

Then the a_i' are

$$u a_1' = a_1 + 2s$$

$$u^2 a_2' = a_2 - s a_1 + 3r - s^2$$

$$u^3 a_3' = a_3 + r a_1 + 2t$$

$\vdots$ (cf. loc. cit.)

Thus we see (fixing π)

Assume $6 \in \mathcal{O}_S^\times$: $\exists!$ x, y s.th. $a_1 = a_2 = a_3 = 0$

namely x', y' with
$$s = -\frac{a_1}{2}$$
$$r = \frac{-a_2 + s a_1 + s^2}{3}$$
$$t = -\frac{a_3 - r a_1}{2}.$$

Cor $6 \in \mathcal{O}_S^\times$, $(E, \pi)/S$. Then $\mathrm{Aut}(E, \pi) = \{id\}$.

Proof x, y these unique elements, $\varphi \in \mathrm{Aut}(E, \pi)$

Then $\varphi^* x$, $\varphi^* y$ reduce to $\varphi^* \pi^{-2} = \pi^{-2}$ resp. π^{-3}

& satisfy same Weierstraß eqn, so $\varphi^* x = x$
$\varphi^* y = y$

by uniqueness. Hence φ extends to $id_{\mathbb{P}^2}$. $\square$

Rmk 1) Shows that
$$\mathrm{Aut}(E) \longrightarrow \mathcal{O}_S^\times$$
$$\varphi \longmapsto [\varphi^* : \omega_E \xrightarrow{\cong} \omega_E]$$
is injective.

2) In char 0, one even has $\mathrm{End}(E) \longrightarrow \mathrm{End}(\omega_E)$ injective.

Proof of representability of $\tilde{\mathcal{M}}[\frac{1}{6}]$

Seen Given π, $\exists!$ way to write

$$E : y^2 = x^3 + ax + b \quad \text{with} \quad \begin{array}{l} x \equiv \pi^{-2} \\ y \equiv \pi^{-3} \end{array}$$

as before.

Conversely, if E, x, y as above

$$\pi := (x \bmod \mathcal{O}_S) / (y \bmod p_* \mathcal{O}_E(2))$$

unique generator of ω making x, y of adapted kind.

Remaining Claim Thus $p^*\pi$ is given by

$$\pi' : \quad -\frac{dx}{2y} = -\frac{dy}{3x^2+a} \quad \text{on } D_+(z).$$

Sketch: Both $p^*\pi$ & π' define generators of $\Omega^1_{E/S}$ and hence differ by element of $\mathcal{O}_S(S)^\times$.

To show: This element equals 1.

May be shown after restriction to $e(S)$.

:) Working locally near $e(S)$, assume $\mathcal{I} = (t)$.

May write $x = \frac{f}{t^2}$, $y = \frac{g}{t^3}$

Then $\pi = (x/y \mod \mathcal{I}^2) = (f/g \cdot t \mod \mathcal{I}^2)$
$= (f/g)(e) \cdot (t \mod \mathcal{I}^2)$.

(The map $\mathcal{I}/\mathcal{I}^2 \xrightarrow{\sim} e^*\Omega^1_{E/S}$ is $\phi \mapsto d\phi$,
so this is $(f/g)(e) \cdot dt \in e^*\Omega^1_{E/S}$.)

:) Now compute $\frac{dx}{-2y}$, which turns out to be defined at ∞.

$$-\frac{dx}{2y} = -\frac{t^2 df - 2tf\,dt}{t^4} \cdot \frac{t^3}{2g}$$

$$= \frac{f}{g}dt - \frac{t}{2g}df \xmapsto{e^*} (f/g)(e)\,dt.$$

∎

§3 Coarse moduli

$F : Sch/S^{op} \longrightarrow Set$ any.

Clear Assume X/S + $[u : X \longrightarrow F] \in F(X)$ are s.th.

$\forall\, Y/S \quad \forall\, v : Y \longrightarrow F \quad \exists!$ factorization w

$$\begin{array}{ccc} Y & & \\ {\scriptstyle \exists!\, w}\downarrow & \searrow^{v} & \\ X & \xrightarrow{\ u\ } & F \end{array}$$

Then u is an iso, i.e. (X, u) represent F.

Too strong for our $F = \mathcal{M}$ (moduli of ECs up to iso).

Idea Consider mapping out property instead:

Def $j : F \longrightarrow X$ coarse moduli space if

1) $$\begin{array}{ccc} F & \xrightarrow{\ j\ } & X \\ & {\scriptstyle \forall}\searrow & \downarrow{\scriptstyle \exists!} \\ & & Y \end{array}$$

2) $j(k) : F(k) \xrightarrow{\ \cong\ } X(k)$ $\forall$ alg closed fields k.

Note Yoneda holds in any category, also Sch/S^{op}, so if F is representable, coarse & fine moduli space exist & coincide.

We proceed as follows: $\tilde{\mathcal{M}} := \tilde{\mathcal{M}}[\frac{1}{6}]$

Recall $\tilde{\mathcal{M}}(S) = \{(E,\pi)/S\}/\cong$

$\mathbb{G}_m$-action $\mu: \mathbb{G}_m \times_{\mathbb{Z}[\frac{1}{6}]} \tilde{\mathcal{M}} \longrightarrow \tilde{\mathcal{M}}$

$(\lambda, (E,\pi)) \longmapsto (E, \lambda\cdot\pi)$.

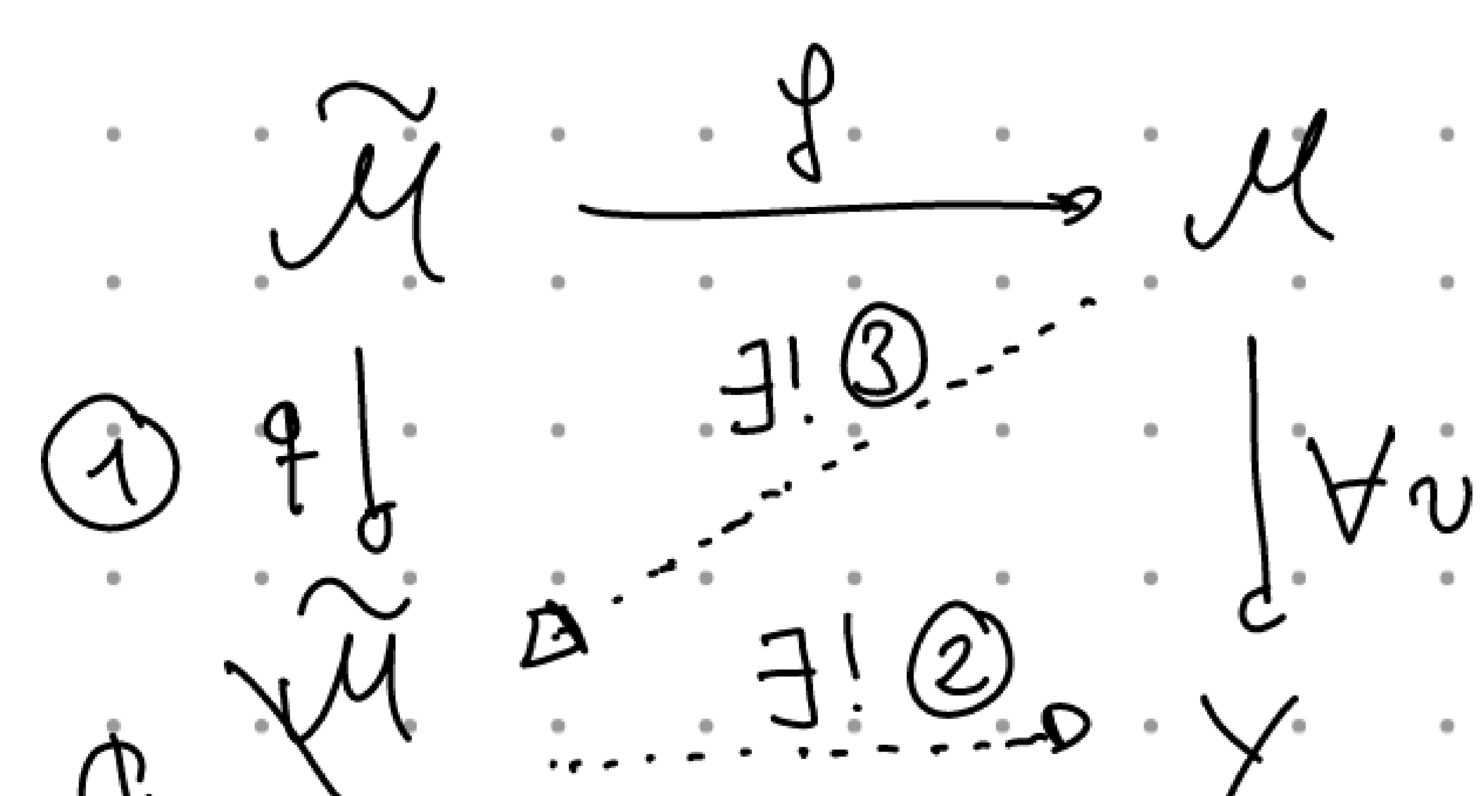
The forget map $\tilde{\mathcal{M}} \xrightarrow{f} \mathcal{M}[\frac{1}{6}]$ is $\mathbb{G}_m$-invariant,

i.e. $f\circ\mu = f\circ p : \mathbb{G}_m \times_{\mathbb{Z}[\frac{1}{6}]} \tilde{\mathcal{M}} \longrightarrow \mathcal{M}[\frac{1}{6}]$.

Thus $\forall\, v : \mathcal{M}[\frac{1}{6}] \longrightarrow Y$, get

$\mathbb{G}_m$-invariant $v\circ f : \tilde{\mathcal{M}} \longrightarrow Y$.

Idea



④ Check $\bar{k}$-condition for ③

① Construct quotient

② Quotient has unique factorization property for $v\circ f$

③ Construct ③ (indep of v) that makes diagram commute.

§ 4 Quotients by $\mathbb{G}_m$

$$X = \operatorname{Spec} A \longrightarrow S = \operatorname{Spec} R \quad \text{integral}$$

$$\left\{ \text{Actions } \mathbb{G}_m \times_S X \xrightarrow{\mu} X \right\} \xleftrightarrow{1:1} \left\{ \text{Gradings } A = \bigoplus_{i \in \mathbb{Z}} A_i \right\}$$

s.th. $A_i A_j \subseteq A_{i+j}$

Recall that an action μ has two axioms:

1) $\left[X \xrightarrow{(1, id)} \mathbb{G}_m \times X \xrightarrow{\mu} X \right] = id_X$

2)
$$\begin{array}{ccc} \mathbb{G}_m \times \mathbb{G}_m \times X & \xrightarrow{(m,\, id_X)} & \mathbb{G}_m \times X \\ {\scriptstyle (id_{\mathbb{G}_m}, \mu)} \downarrow & \circlearrowleft & \downarrow {\scriptstyle \mu} \\ \mathbb{G}_m \times X & \xrightarrow{\mu} & X \end{array}$$
associativity.

Here is how to construct grading from action.

Say $\mu \rightsquigarrow \mu^* : A \longrightarrow R[t^{\pm 1}] \otimes_R A$

$$a \longmapsto \sum_i t^i \otimes a_i$$

Claim Each a_i satisfies $\mu^*(a_i) = t^i \otimes a_i$

Proof Associativity of $\mathbb{G}_m$-action means for a

$$\sum_i t^i \otimes t^i \otimes a_i = \sum_i \sum_j t^i \otimes t^j \otimes (a_i)_j$$

Since $t^i \otimes t^j$ R-basis for $R[t^{\pm 1} \otimes 1, 1 \otimes t^{\pm 1}]$, means
$$(a_i)_j = \begin{cases} a_i & i = j \\ 0 & i \neq j \end{cases} \qquad \square$$

Put $A_i := \{ a \in A \text{ s.th. } \mu^*(a) = t^i \otimes a \}$

Then $a_i \in A_i$ by above claim.

Claim $A = \bigoplus_{i \in \mathbb{Z}} A_i$.

Proof $\mu \circ (1, id_X) = id_X$ translates to

$$a \overset{\mu^*}{\longmapsto} \sum t^i \otimes a_i \overset{t \mapsto 1}{\longmapsto} \sum a_i \overset{!}{=} a \qquad \square$$

The property $A_i \cdot A_j \subset A_{i+j}$ is because μ^* is a ring map.

Now let $T = \operatorname{Spec} B$ any, $v : X \to T$.

Then v is $\mathbb{G}_m$-invariant $\iff$ $v^* : B \to A$ factors $\supseteq A_0$.

Thus $X \xrightarrow{q} Y := \operatorname{Spec} A_0$ is categorical quotient in affine S-schemes.

Prop Assume X, Y noetherian. Then q is a categorical quotient in all S-schemes.

Proof 1) Given $V \subseteq Y$ open, $U = q^{-1}(X) \subseteq X$ is open + $\mathbb{G}_m$-stable.

$\rightsquigarrow$ Get $\mathbb{G}_m \times U \xrightarrow{\mu} U$

$$(q_* \mathcal{O}_X)^G(V) := \{ f \in \underset{\mathcal{O}_X(U)}{\underset{\shortparallel}{(q_* \mathcal{O}_X)(V)}} \mid \mu^* f = p^* f \}.$$

Define subsheaf $(q_* \mathcal{O}_X)^G \subseteq q_* \mathcal{O}_X$.
(Because $\mu^* f = p^* f$ may be checked locally on U.)

q $\mathbb{G}_m$-invariant $\Rightarrow$ $\mathcal{O}_Y \xrightarrow{q^*} q_* \mathcal{O}_X$ factors $\dashrightarrow$ $(q_* \mathcal{O}_X)^G \subseteq q_* \mathcal{O}_X$

<u>Claim</u> $\mathcal{O}_Y \xrightarrow{\cong} (q_* \mathcal{O}_X)^G$.

<u>Proof</u> Already have map of sheaves, so may check locally, e.g. on affines.

But $\forall\ A_0 \longrightarrow C_0$,

$$(C_0 \otimes_{A_0} A)_0 = C_0 \quad . \quad \square \text{ Claim.}$$

2) Let $Z = V(I) \subseteq X$ any closed, consider

$$\begin{array}{ccc} \mathbb{G}_m \times Z & \overset{?}{\dashrightarrow} & Z \\ \downarrow & & \downarrow \\ \mathbb{G}_m \times X & \xrightarrow{\ \mu\ } & X \end{array}$$

Z $\mathbb{G}_m$-stable $\overset{=}{\text{def}}$ factorization $?$ exists.
(Then $\mathbb{G}_m \curvearrowright Z$.)

Z $\mathbb{G}_m$-stable $\iff$ Any $a \in I$ maps to 0 in

$$a \overset{\mu^*}{\longmapsto} \sum t^i \otimes a_i \longmapsto \sum t^i \otimes (a_i \bmod I)$$

$$\iff \Big(a \in I \implies \text{all } a_i \in I \Big)$$

i.e. I homogeneous.

Consequence For I_i family of invariant ideals,

$$\Big(\sum I_i\Big) \cap A_0 = \sum (I_i \cap A_0)$$

Reformulation $\quad \text{Closure}\Big(q\big(\bigcap_i Z_i\big)\Big) = \bigcap_i \text{Closure}\big(q(Z_i)\big)$

for invariant Z_i.

3) <u>Claim</u> $X \xrightarrow{q} Y$ is surjective.

<u>Proof</u> Assume $\mathfrak{p} \in \operatorname{Spec} A_0$ and

$$b = \sum_k \sum_{j \in J_k} a_k^{(j)} p_k^{(j)} \in A_0 \cap \mathfrak{p} \cdot A,$$

where $\deg(a_k^{(j)}) = k$ for all k, j.

Then $b = \sum_{j \in J_0} a_0^{(j)} p_0^{(j)} \in \mathfrak{p}$ already

by using the grading property.

So $\mathfrak{p} = A_0 \cap \mathfrak{p}A$.

A noetherian $\Rightarrow$ $A/\mathfrak{p}A$ has fin many prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ only.

Then $\mathfrak{q}_i \cap A_0 = \mathfrak{p}$ for some i because

$$\left(\bigcap_{i=1}^{r} \mathfrak{q}_i\right)^n \subset \mathfrak{p}A \subset \bigcap_{i=1}^{r} \mathfrak{q}_i \quad \text{for } n \gg 0.$$

Claim $Z \subseteq X$ closed + $\mathbb{G}_m$-stable.

Then $q(Z)$ is closed.

Proof Let $y \in \overline{q(Z)} \setminus q(Z)$ any point.

$Y := \overline{\{y\}}$ closure.

Then $q^{-1}(Y) \subseteq X$ closed $\mathbb{G}_m$-stable.

(all with reduced scheme structure, say)

We get $Y = \overline{q(Z)} \cap Y$

surjectivity $\to$ $= \overline{q(Z)} \cap q(q^{-1}(Y))$

$\overset{2)}{=} \overline{q(Z \cap q^{-1}(Y))}$ $(*)$

X noetherian $\Rightarrow$ $Z \cap q^{-1}(Y)$ has fin. many

generic points, say $z_1, \ldots, z_r$.

Then $(*)$ equals $\bigcup \overline{q(z_i)}$.

Since $y \notin q(Z)$, this union does not contain y !

$\Rightarrow$ $\overline{q(Z)} \setminus q(Z) = \emptyset$. $\square$ claim.

4) <u>End of proof</u>: $X \xrightarrow{v} T$ $\mathbb{G}_m$-invariant.

$W \subseteq T$ open affine $\longrightarrow$ $v^{-1}(W) \subset X$ open + $\mathbb{G}_m$-stable

Thus $Z = X \setminus v^{-1}(W)$ + reduced scheme str

is closed + $\mathbb{G}_m$-stable

3) $\longrightarrow$ $q(Z)$ closed, so $V = Y \setminus q(Z)$

is open with $q^{-1}(V) \subseteq v^{-1}(W)$.

Covering V by affines + using 1), we find a unique $V \longrightarrow W$ that fits into

$$v^{-1}(W) \supseteq q^{-1}(V) \xrightarrow{q} V, \quad q^{-1}(V) \xrightarrow{v} W, \quad V \longrightarrow W$$

Varying W, get $\bigcup_{\text{all occurring } V} V \longrightarrow T$

(The gluing comes from step 1).)

<u>Final argument</u>: Varying W, $v^{-1}(W)$ cover X.

Hence $\bigcap_W (Y \setminus V) = \bigcap_W q(X \setminus v^{-1}(W))$

$$\overset{2)+3)}{=} q\left(\bigcap_W X \setminus v^{-1}(W)\right)$$

$$= q(\emptyset) = \emptyset. \qquad \blacksquare$$

<u>Rmk</u> Proof works for reductive groups acting on affine schemes, at least when $S = \operatorname{Spec} k$.

see [Mumford GIT].

<u>Examples</u> $S = \operatorname{Spec} k$

1) $\mathbb{G}_m \curvearrowright \mathbb{A}^n$, $\lambda \cdot (x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$.

$\longleftrightarrow$ $k[T_1, \ldots, T_n]$ w/ $\deg(T_i) = 1 \ \forall i$.

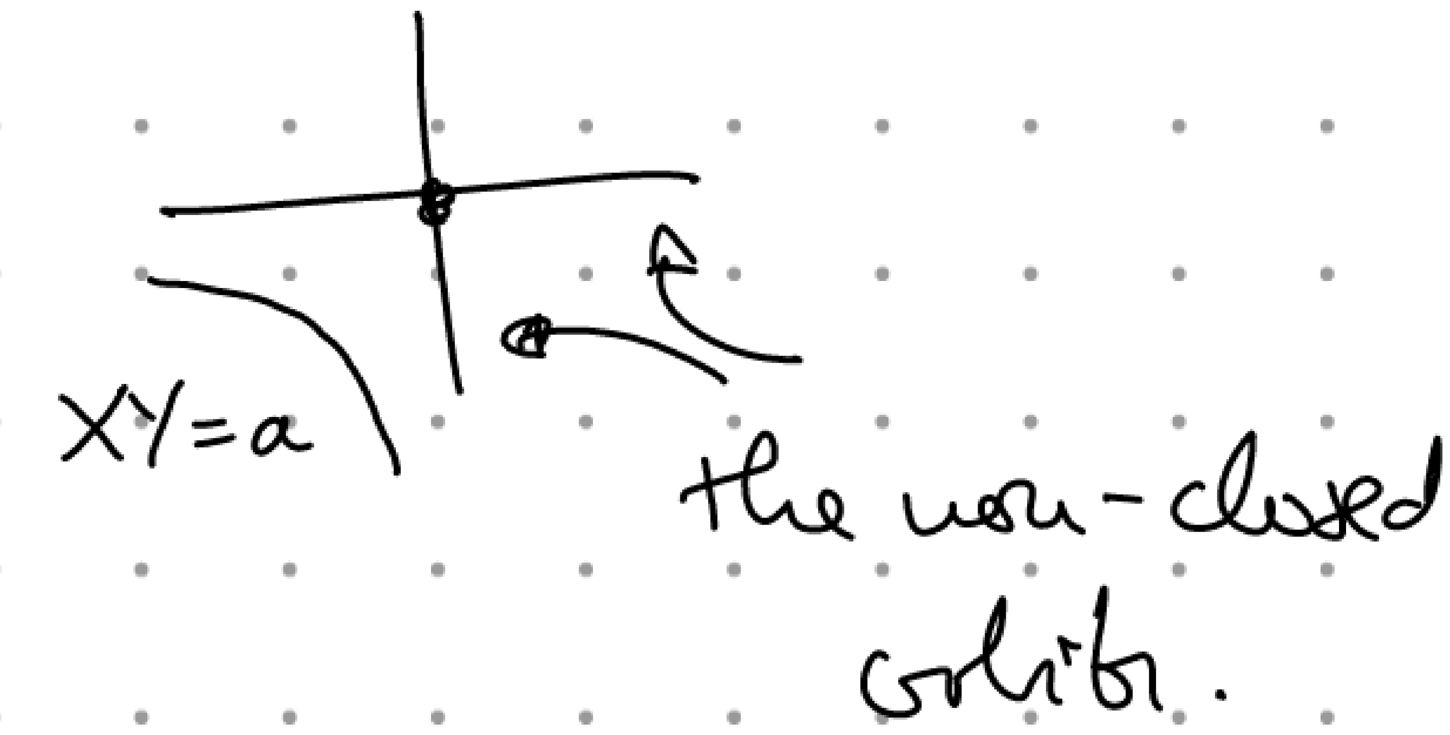
Then $\mathbb{A}^n /\!\!/ \mathbb{G}_m = \operatorname{Spec} k$.

<u>Reason</u> All orbits contain $\{0\}$ in closure. So any $\mathbb{G}_m$-invariant function on $\mathbb{A}^n$ is constant.

2) $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ $\lambda(x, y) = (\lambda x, \lambda^{-1} y)$.

Then $\mathbb{G}_m \backslash \mathbb{A}^2 = \operatorname{Spec} k[XY]$.

$q^{-1}(0) = V(XY) \subseteq \mathbb{A}^2$



$q^{-1}(a)$ w/ $a \neq 0$

is hyperbola $XY = a$, a closed orbit.

§5 Application to $\widetilde{\mathcal{M}}$

Recall $\widetilde{\mathcal{M}} = \operatorname{Spec} R$, $R = \mathbb{Z}[\frac{1}{6}][a_4, a_6][\Delta^{-1}]$

$$\Delta = 4a_4^3 + 27a_6^2$$

$$\mathcal{E} : \quad y^2 = x^3 + a_4 x + a_6 \qquad (*)$$

$$\pi = \frac{-dx}{2y}$$

Seen last time: $\pi' = u\pi$

$$\Rightarrow x' = u^{-2}x, \quad y' = u^{-3}y.$$

Thus, multiplying $(*)$ by u^{-6},

$$a_i \longmapsto u^{-i} a_i$$

Thus $\mathbb{G}_m \curvearrowright \tilde{\mathcal{M}}$ $\Rightarrow$ $\deg(a_4) = 4$

$\deg(a_6) = 6.$

$\Rightarrow$ $\deg(\Delta) = 12.$ (At least after switching the sign from my convention.)

Then (check!):

$$R_0 = \mathbb{Z}[\tfrac{1}{6}][\tfrac{a_4^3}{\Delta}] = \mathbb{Z}[\tfrac{1}{6}][j]$$

$$\Delta = -16(4a_4^3 + 27a_6^2)$$

$$j = -1728 \frac{4a_4^3}{\Delta}$$

Intrinsic defn of j-invariant! (Up to constants $\in \mathbb{Z}[\tfrac{1}{6}]^\times$)

$$\Rightarrow j : {}_{\mathbb{G}_m}\backslash\tilde{\mathcal{M}} \xrightarrow{\cong} \mathbb{A}^1_{\mathbb{Z}[\frac{1}{6}]}$$

Proof of coarse moduli property:

Define $\mathcal{M}[\tfrac{1}{6}] \to \mathbb{A}^1_{\mathbb{Z}[\frac{1}{6}]}$ through j invariant:

Given $E \in \mathcal{M}(S)$, trivialize $\omega_{E/S}$ locally on $S = \cup U_i$, pick Weierstraß equ, take j-invariants $j \in \mathcal{O}_S(U_i)$.

Since j-invariant is indep of Weierstraß eqn,

glues to section $j(E) \in \mathcal{O}_S$. Moreover,

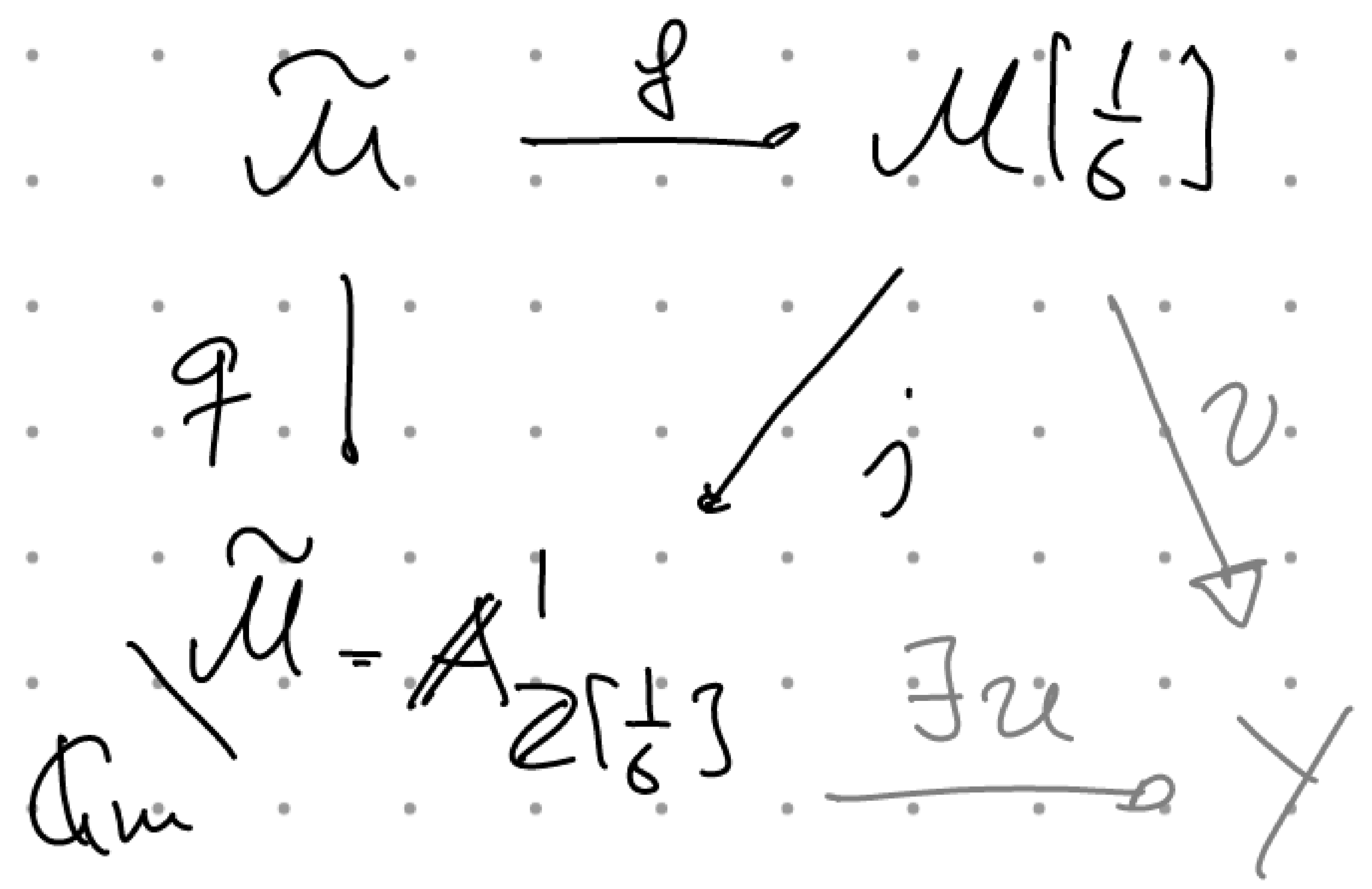
(Base change of Weiersh. Eq is

Weierstraß Eq w/ pullback coefficients)

$\Longrightarrow \quad j(T \times_S E) = u^* j(E) \in \mathcal{O}_T(T) \qquad u: T \to S$.

Thus $E \mapsto j(E)$ is natural transformation.

<u>Categorical property</u>



Black triangle commutes by construction.

Given v, $v \circ f : \widetilde{\mathcal{M}} \to Y$ is $\mathbb{G}_m$-invariant,

so factors in unique way through j.

This defines u. Still to check: $v = u \circ j$

Note that $\mathcal{M} : Sch/S^{op} \longrightarrow Set$ is just a functor (a.k.a. presheaf). We need to see

$\forall\ S \xrightarrow{\ \alpha\ } \mathcal{M}$, the compositions agree,

$\alpha \longleftrightarrow E/S$ $\qquad\qquad v \circ \alpha = u \circ j \circ \alpha$

<u>Idea</u> Factor locally through $\widetilde{\mathcal{M}}$.

Zariski locally on S, say on $S = \cup\, U_i$,

$\exists$ lift $\tilde{\alpha}_i : U_i \longrightarrow \widetilde{\mathcal{M}}$

$$\begin{array}{ccc} U_i & \xrightarrow{\tilde{\alpha}_i} & \widetilde{\mathcal{M}} \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{M} & \end{array}$$

(trivialize $\omega_{E/S}$ locally for this)

Thus
$$v \circ \alpha|_{U_i} = v \circ f \circ \tilde{\alpha}_i$$
$$= u \circ j \circ f \circ \tilde{\alpha}_i$$
$$= u \circ j \circ \alpha|_{U_i}$$

The property $j(k) : \mathcal{M}(k) \cong k$ is in [Silverman Prop 1.4]. $\blacksquare$

<u>Fun observation</u> $\nexists$ $\mathcal{E} \to \mathbb{A}^1_{\mathbb{Z}[\frac{1}{6}]}$ of j-invariant j.

Namely $\mathbb{A}^1_{\mathbb{Q}}$ is a PID, so

$\omega_{\mathcal{E}|\mathbb{A}^1_{\mathbb{Q}}}$ would be trivial. Then

$\Delta(\mathcal{E}|_{\mathbb{A}^1_{\mathbb{Q}}}) \in \mathbb{Q}[j]^\times = \mathbb{Q}^\times$ and the

solution $\dfrac{a_4(j)^3}{\Delta} = j$ has no solution in $\mathbb{Q}[j]$.

This again shows that $\mathcal{M}$ has no fine moduli space.

<u>Rmk</u> The property of a coarse moduli space to have a section $F \rightleftarrows$ (Coarse for F)

is <u>not</u> sufficient to F being representable.

<u>E.g.</u> $[\mathrm{Pic} : S \mapsto \mathrm{Pic}(S)] \longrightarrow \operatorname{Spec} \mathbb{Z}$

is coarse moduli & $\exists$ a line bundle on $\operatorname{Spec} \mathbb{Z}$.